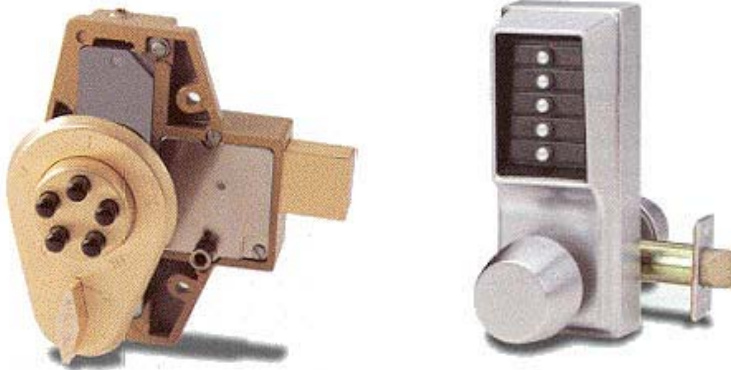


A Pushbutton Lock

Here's a problem that will weave its way in and out of many of the topics we discuss this summer. We'll start working on it right now, but don't feel that you have to get the "right answer" right away. We'll keep coming back to it, even after you might be satisfied that you know what's going on. There's a lot to discover in this problem.

Several companies make a combination lock that is used in many public buildings. It comes in several versions. Here are two:



These 5-button devices are purely mechanical (no electronics). You can set the combination using the following rules:

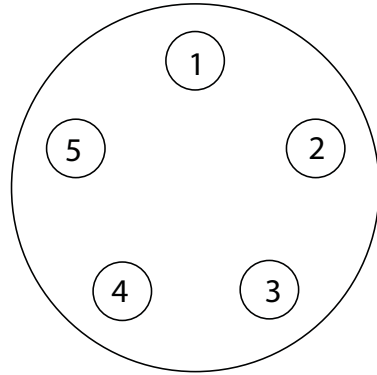
1. A combination is a sequence of 0 or more pushes, each push involving at least one button.
2. Each button may be used at most once (once you press it, it stays in).
3. Each push may include any of the buttons that haven't been pushed yet, up to and including all remaining buttons.
4. The combination does *not* need to include all buttons.
5. When two or more buttons are pushed at the same time, order doesn't matter.

There's one possible no-push combination: the door's just already unlocked. Not a great combination, but it counts.

Here are some possible combinations:

- {{1, 3}, {4}}
- {{1, 2, 4}, {3, 5}}
- {{3}, {1, 2}}
- {{1, 2}, {3}}
- {{1, 2, 3, 4, 5}}
- {}
- {{2}, {1}, {3}}
- {{1, 2}, {4}, {3, 5}}
- {2}

Notation: {{ 1,2}, {3}} means "press 1 and 2 together, then press 3."



An artist's rendition of a pushbutton lock.

At least one company advertises that there are thousands of combinations, and the question is, "Is the company telling the truth?"

PROBLEM

How many combinations are there on a 5-button lock?

1 Trains

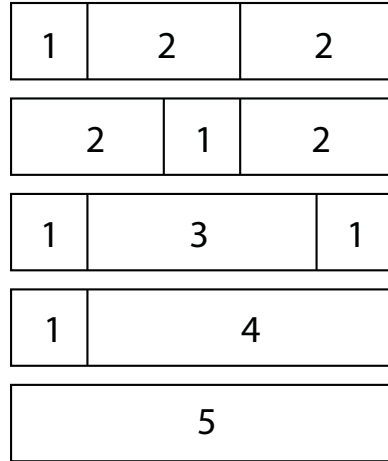
Hey, welcome to the class. We know you'll learn a lot of math here—maybe some new tricks, maybe some new perspectives on things you're already familiar with. A few things you should know about how the class is organized:

- You do not have to answer all of the questions, ever. If you're answering every question, we haven't written the problem sets correctly.
- You are not required to get to a certain problem number. Some participants might spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- Have fun! Make sure you're spending time working on problems you're interested in. Feel free to skip problems that you're already sure of. Relax and enjoy!
- Each day starts with problems like the ones below, intended to be picked up on regardless of how much or little work you've done on prior sets. Try them as a starting point.
- *Whatever you do, do well.* Flying through the problem set helps no one, especially yourself—you're going to miss the big ideas that others are grabbing onto! There is more to be found in these problems than their answers.

Okay, so let's get started. The first set of problems revolve around “trains” of rods. You can use rods of integer sizes to build “trains” that all share a common length.

A “train of length 5” is a row of rods whose combined length is 5. Here are some examples:

Think of the rods as Cuisenaire Rods if you know what these are. Unless you are told otherwise, you have an unlimited supply of all the rod types.



Notice that the 1-2-2 train and the 2-1-2 train contain the same rods but are listed separately. If you use identical rods in a different order, this is a separate train.

1. How many trains of length 4 are there?
2. How many trains of length 5 are there?
3. Find a formula for the number of trains of length n . Come up with a convincing reason that your rule is correct.
4. Create an algorithm that will generate all the trains of length n .
5. How many trains of length n are there that use *only* cars of length 1 and 2? Find a general rule, and explain why your rule works.
6. How many trains of length 11 are there that use only cars of length 1, 2, and 3?

Assume you have rods of every possible integer length available.

Neat Stuff

This section includes a variety of problems each day, which range from practice to extension. Pick and choose problems to work on, depending on your background and focus. Don't be surprised to see problems here repeated in later sets; that's our way of suggesting you check it out sometime.

7. Suppose there are three flavors of ice cream: pistachio, strawberry, and chocolate. How many different three-scoop cones can you make using each of these flavors exactly once?

Note that in a *cone*, it is important which scoop is on top. Thus, a pistachio-strawberry-chocolate cone is different from a strawberry-chocolate-pistachio cone.

8. Suppose you want a four-scoop cone with one scoop each of pistachio, strawberry, chocolate, and butter pecan. After your release from the mental hospital, how many different cones could you make with these flavors? Explain your reasoning carefully.
9. (a) How many different cones can you make from 5 scoops of different flavors?
 (b) How many different cones can you make from n scoops of different flavors?
10. Ben & Jerry's serves 36 different flavors of ice cream.
 (a) How many different three-scoop cones can you make at Ben & Jerry's?
 (b) How many different four-scoop cones can you make at Ben & Jerry's?
 (c) Find a rule for determining the number of different cones available at Ben & Jerry's in terms of the number of scoops on the cone.
11. In a *bowl* of ice cream, the order of the scoops does not matter. Therefore, a chocolate-vanilla bowl is the same as a vanilla-chocolate bowl. If you've taught the IMP *Cones and Bowls* unit, you probably saw this one coming. . .
 (a) At a certain ice cream shop, you can make 465 different two-scoop bowls of ice cream. How many different two-scoop cones can you make? Explain how you know.
 (b) Find the number of flavors offered at this ice cream shop.
12. If you can make 220 different three-scoop bowls of ice cream, how many different three-scoop cones can you make?
13. (a) If you can make 210 different four-scoop bowls of ice cream, how many different four-scoop cones can you make?
 (b) If you can make 3024 different four-scoop *cones* of ice cream, how many different four-scoop *bowls* can you make?
14. If you can make 55440 different five-scoop cones, how many different five-scoop bowls can you make?
15. (a) At Ben & Jerry's (where there are 36 flavors), how many different five-scoop bowls can be made?
 (b) Find a rule for determining the number of different bowls of k scoops at Ben & Jerry's.

Tough Stuff

This section has some difficult problems! Try these if you're up for a challenge or already feel pretty confident about the problems in the rest of the set. Our guarantee: something challenging every day.

16. If you made all the trains of length 5, how many cars of length 1 were used? length 2, 3, 4, 5? See if you can find a general rule for the number of cars of length k you'd need to make all the trains of length n .
17. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?
18. How many three-scoop bowls could you make at Ben & Jerry's if you *were* allowed to duplicate flavors? Is there a general rule at work?

2 More Trains

We thought things went very well in the last session. However, we do have a few general comments.

We thought of naming this set “Trains 2: Electric Boogaloo”, but thought better of it.

- **Modeling.** Everyone is very good at recognizing patterns in numbers. However, sometimes it is a good idea to put yourself into the problem. For example, suppose you are a switch agent who has to put cars together to make a train. How would you go about it? Where would you start?
- **Why?** You may have noticed that we asked this question often. When you notice a pattern in numerical data, it is natural to make a conjecture about a general rule. But how do you know that this is not just a coincidence? How do you know that the pattern is true for all cases.
- **The lock problem.** Remember this? Does anything you have learned so far help you to analyze the lock?

Here are today’s problems. A couple are repeats from last time. If you solved them then, you do not have to do it again.

1. How many trains of length 10 are there that use cars of lengths 1 and 2 only?
2. How many trains of length n are there that are combinations of cars of lengths 1 and 2 only? Find a general rule, and explain why your rule works.
3. How many trains of length 11 are there that use cars of lengths 1, 2, and 3 only?
4. How many trains of length 6 are there?
5. How many trains of length 6 are there that use exactly one car? two cars? three cars?
6. Celeste shows you a full chart of all the trains of length 4. Try to figure out a way to use the chart of trains of length 4 to generate all the trains of length 5.
7. Using cars of lengths 2 and 3 only, how many trains of length 11 can be made?

Hm, supposedly there are twice as many trains of length 5...

8. Make a table of how many trains of length n can be made using cars of length 2 and 3 only, for n from 1 to 11. Is there a rule you could use to continue the table?

Neat Stuff.

These problems may look familiar, since they are the same as last time!

9. Suppose there are three flavors of ice cream: pistachio, strawberry, and chocolate. How many different three-scoop cones can you make using each of these flavors exactly once?
10. Suppose you want a four-scoop cone with one scoop each of pistachio, strawberry, chocolate, and butter pecan. After your release from the mental hospital, how many different cones could you make with these flavors? Explain your reasoning carefully.
11. (a) How many different cones can you make from 5 scoops of different flavors?
 (b) How many different cones can you make from n scoops of different flavors?
12. Does problem 11 generally get easier or harder if you're allowed to repeat flavors within a cone? Why?
13. Ben & Jerry's serves 36 different flavors of ice cream.
 (a) How many different three-scoop cones can you make at Ben & Jerry's, if you never use the same flavor twice?
 (b) How many different four-scoop cones can you make at Ben & Jerry's, if you never use the same flavor twice?
 (c) Describe a rule for determining the number of different cones available at Ben & Jerry's in terms of the number of scoops on the cone.
14. In a *bowl* of ice cream, the order of the scoops does not matter. Therefore, a chocolate-vanilla bowl is the same as a vanilla-chocolate bowl.
 (a) At a certain ice cream shop, you can make 465 different two-scoop bowls of ice cream. How many different two-scoop cones can you make? Explain how you know.

Note that in a *cone*, it is important which scoop is on top. Thus, a pistachio-strawberry-chocolate cone is different from a strawberry-chocolate-pistachio cone.

It's like the lock: pushing #4 and then #2 is like a cone; order matters. Pushing *both* #4 and #2 at once is like a bowl.

- (b) Find the number of flavors offered at this ice cream shop.
15. If you can make 220 different three-scoop bowls of ice cream, how many different three-scoop cones can you make?
16. (a) If you can make 210 different four-scoop bowls of ice cream, how many different four-scoop cones can you make?
- (b) If you can make 3024 different four-scoop *cones* of ice cream, how many different four-scoop *bowls* can you make?
17. If you can make 55440 different five-scoop cones, how many different five-scoop bowls can you make? Please, a five-scoop bowl should only be ordered for mathematical purposes.
18. (a) At Ben & Jerry's (where there are 36 flavors), how many different five-scoop bowls can be made?
- (b) Find a rule for determining the number of different bowls of k scoops at Ben & Jerry's.
19. Does problem 18 generally get easier or harder if you're allowed to repeat flavors within a bowl? Why?

Tough Stuff.

20. If you made all the trains of length 5, how many cars of length 1 were used? length 2, 3, 4, 5? See if you can find a general rule for the number of cars of length k you'd need to make all the trains of length n .
21. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?
22. In a coin-flipping game, you flip a fair coin (heads or tails) ten times. If you flip heads twice in a row at any point during the game, you lose. Find the probability that you win at this game.
23. How many three-scoop bowls could you make at Ben & Jerry's if you *were* allowed to duplicate flavors? Is there a general rule at work?

3 Even More Trains

Or, Trains 3: Judgment Day.

You might look back at the lock problem to see if you have any new ideas. Here are today's problems.

- How many trains of length 7 are there?
- (a) Make a table for $n = 2$ to 8 for the number of trains of length n that use *exactly* two cars.
(b) Repeat for *exactly* three cars.
- Complete this table, with the train length on the vertical and the number of cars on the horizontal:

	# Cars Used						
	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1		1				
4	1			1			
5	1				1		
6	1	5	10			1	
7	1						1

- Physically make all the trains of length 6 that use exactly two cars, and (separately, without destroying the two-car trains) all the trains of length 6 that use exactly three cars.
- Can you think of a way to use the trains you built in problem 4 to make all the trains of length 7 that use exactly three cars?
- You've got an unlimited supply of cars 2, 3, and 5. How many different trains of length 12 can you make with all these cars?
- How many trains of length 10 can you make with *no* cars of length 1?
- Make a table of how many trains of length n can be made using cars of length 2 and 3 only, for n from 1 to 11. Is there a rule you could use to continue the table?

In other words, use only cars of length 2, 3, 4, or more.

Neat and/or Needed Stuff

9. At the world's largest ice cream store, you can order 3,464,840 bowls of ice cream with four different-flavored scoops. *Without* figuring out how many flavors there are, can you describe a way to find how many four-scoop cones of ice cream are available?
10. (a) Suppose you can make 156 different two-scoop cones at a certain ice cream shop. How many different flavors does this shop offer?
- (b) Suppose you can make 2730 different three-scoop cones at a certain ice cream shop. How many different flavors does this shop offer?
11. Let n and r be non-negative integers with $n \geq r$. Define

$${}_n P_r = \frac{n!}{(n-r)!}$$

Explain why ${}_n P_r$ is the number of *permutations* (or, cones) of n things taken r at a time.

12. Using the definition above, find ${}_n P_0$. Explain using the ice cream analogy.
13. What's a simpler rule for ${}_n P_n$?
14. Let n and r be non-negative integers with $n \geq r$. Define

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Explain why $\binom{n}{r}$ is the number of r -scoop bowls you can make with n scoops of ice cream.

15. Using the definition, find $\binom{n}{0}$. Explain using the ice cream analogy.
16. What's a simpler rule for $\binom{n}{n}$? Explain using ice cream. Mmmm, fattening.
17. A pizza store offers 10 kinds of toppings.
- (a) Suppose you want exactly 2 toppings on your pizza. How many different pizzas can you make?
- (b) Suppose you want exactly 8 toppings on your pizza. How many different pizzas can you make?

By the way, $0!$ is 1. We'll talk about some convincing reasons why this makes sense.

We sometimes write ${}_n C_r$ instead of $\binom{n}{r}$.

On calculators, find $\binom{n}{r}$ by using the ${}_n C_r$ function. On the TI-84, type the first number, then find "nCr" under the MATH > PRB menu. On the TI-89, it is "nCr(n, r)" so you'll need to grab "nCr" under the MATH > PRB menu first. Or type it in. Teachers using both calculators have noted that "nCr" is often easier to find under CATALOG, since it is the first entry starting with the letter n .

- (c) What's going on here?
- (d) Explain why $\binom{n}{r} = \binom{n}{n-r}$.
18. Using the definition above, find $\binom{12}{0}$ and $\binom{12}{12}$. Explain the results using the concept of pizza toppings.
19. A 13-card hand is dealt from a standard 52-card deck. Find the probability that this hand contains exactly 3 aces and exactly 2 kings (which means exactly 8 of the rest...).

Tough Stuff

20. If you made all the trains of length 5, how many cars of length 1 were used? length 2, 3, 4, 5? See if you can find a general rule for the number of cars of length k you'd need to make all the trains of length n .
21. What's the *average* (mean) length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?
22. In a coin-flipping game, you flip a fair coin (heads or tails) ten times. If you flip heads twice in a row at any point during the game, you lose. Find the probability that you win at this game.
23. A binary string of length 12 has twelve digits: all are ones or zeros. One example is

011010011100

How many 12-digit binary strings...

- (a) ... do not start or end with a 1, *and*
- (b) ... do not include any two consecutive ones?

4 Training Day

We are seeing a lot of different ways to approach the lock problem. Here are some thoughts.

- A lot of your problems can be solved by inductive reasoning. This amounts to counting things by relating the problem to the counts of smaller items of the same type. Can you see how this will help with the lock problem?
- How do all of these train problems help to understand the lock problem. Can you see any connection?

Here are today's problems.

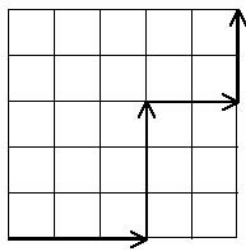
1. Suppose you have an unlimited supply of cars of lengths 3, 4, and 8 only. Find the total number of ways to make a train of length 14, using any method you like.
2. Suppose you have an unlimited supply of cars of lengths 4 and 7 only. Are there any train lengths you *can't* make? Which ones?
3. Suppose you have an unlimited supply of cars of lengths 2 and 4 only. Are there any train lengths you *can't* make? Which ones?
4. If you were allowed to write the letters of the word MINIMUM in any order, how many total different seven-letter "words" could you make?
5. One way to make a train of length 14 is to use three cars of length 1, two cars of length 2, one car of length 3, and one car of length 4. How many different-looking trains of length 14 could you make using these specific cars?
6. The company SW INC ships wolves and sheep. Naturally, the wolves and the sheep cannot be shipped in the same rail cars. Suppose that rail cars of all lengths are available. How many trains of length 7 are possible, assuming that sheep cars and wolf cars are considered to be different?

One of these "words" is INMUMIM.

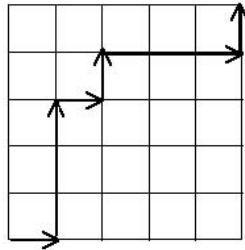
Neat and Useful Stuff

7. Using either factorial or “choose” notation, write an expression for the number of ways to reorder the letters of 70’s singing sensation ABBA.
8. Repeat problem 7 for 80’s singing sensation BANANARAMA.

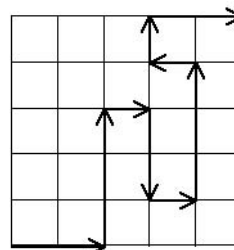
Ms. D’Amato likes to take a different route to work every day. She will quit her job the day she has to repeat her route. Her home and work are pictured in the grid of streets below.



A valid trip



Another valid trip



Not a valid trip

9. If Ms. D’Amato never backtracks (she only travels north or east), how many days will she work at this job?
10. How many more days can Ms. D’Amato work if she moves two blocks further away? Does it matter in which direction she moves?
11. Hey, what’s the sum of the numbers in the fifth row of Pascal’s Triangle? (Okay, so it’s not a hard question.)
12. Hey, what’s the sum of the *squares* of the numbers in the fifth row of Pascal’s Triangle? (See, a little harder.)
13. There are 12 students on a committee; 8 are juniors, and 4 are seniors. Determine the number of ways of forming the following subcommittees.
 - (a) A subcommittee of 5 juniors.
 - (b) A subcommittee of 3 seniors.
 - (c) A subcommittee of 6 students.
 - (d) A subcommittee of 3 juniors and 2 seniors.

The fifth row of Pascal’s Triangle starts 1, 5, ...

14. Jimmy is in a class of 20 students. If a 4 person committee is chosen at random to represent the class, how many possible committees are there? How many will include Jimmy? Use two different methods to figure out how many will exclude Jimmy?
15. A jar contains 3 white balls and 4 red balls. 2 balls are drawn. Find the probability that both balls are the same color. It may help you to...
- (a) Find the number of ways to draw 2 balls from the jar.
 - (b) Find the number of ways to draw 2 white balls.
 - (c) Find the number of ways to draw 2 red balls.
16. Find $P(\text{same color})$ from problem 15 using a tree diagram. Which method do you prefer?
17. (a) A committee of three is chosen from 3 Republicans and 4 Democrats. What is the probability that the committee consists of 2 Republicans and 1 Democrat?
- (b) Verify your result in (a) using a tree diagram, where the committee members are selected one at a time.

$P(A)$ means the probability of event A happening. Here, the "event" is drawing two balls of the same color.

Well... assume for the sake of this problem that the selection is random.

Tough Stuff

18. With 9 people, committees can be picked with as few as 0 people and as many as 9. Give a convincing argument (or, to use the vernacular, *prove*) that the total number of committees that can be formed with an odd number of members is the same as the number of committees with an even number of members.
19. With 8 people, committees can be picked with as few as 0 people and as many as 8. Give a convincing argument that the total number of committees that can be formed with an odd number of members is the same as the number of committees with an even number of members.
20. How many odd numbers are there in the 100th row of Pascal's Triangle?

5 Ben & Jerry's Day

On Ben & Jerry's Day we feature a Ben & Jerry's problem. Suppose that Ben & Jerry's brings us three flavors of ice cream (perhaps vanilla (V), chocolate(C), and strawberry(S)) in order to provide dessert for everyone. Unfortunately, there are rules:

- A dessert is a sequence of 0 or more bowls of ice cream.
- A dessert may contain at most one scoop of each flavor, no matter how many bowls are used.
- A bowl must contain at least one scoop of ice cream, and may contain scoops of any flavors not already sampled.
- A dessert does not have to include all of the flavors.

Here are some possible desserts:

- | | | |
|-----------------|------------|---------------|
| • { } | • {{C, V}} | • {{C, V, S}} |
| • {{S, V}, {C}} | • {{S}} | • {{V}, {S}} |

In the above table, the notation $\{\{C\}, \{V\}\}$ means a bowl with one scoop of chocolate followed by a bowl with one scoop of vanilla.

1. How many different desserts are possible?
2. If you are having trouble with three flavors, suppose that Ben & Jerry's brought only two flavors. Using the same rules, how many desserts are possible?

We have to have some problems that are not ice cream related, so here goes.

3. Without a calculator, expand $(h + t)^5$.
4. You flip a coin five times. How many ways are there to flip two heads and three tails?
5. Spend at least 15 minutes thinking about the lock problem.

In our next and final session next Monday we will wrap things up. We will have presentations of solutions, both partial and complete. In particular we will present solutions of the lock problem, so be thinking about that. We will also provide a final list of problems in case you want something to do the rest of the summer.

Rumor has it that these rules are set by a mysterious organization that is identified with a four letter acronym, the last two letters of which are SD.

Neat and Useful Stuff

6. How many “words” can you make from PAPAKONSTANTINO?
7. If you randomly rearrange the letters in PAPAKONSTANTINO, find the probability that ...
 - (a) the first letter is A.
 - (b) the first *two* letters are A.
 - (c) the first four letters are A.
 - (d) the letters spell out KAPPASANTONUTION.
8. A bag contains 16 T-shirts of which 6 have red stripes. Two T-shirts are randomly drawn from the bag.
 - (a) Find the probability that neither T-shirt has red stripes.
 - (b) Verify your result using a tree diagram.
 - (c) Another bag contains 8 T-shirts of which 3 have red stripes. Is the probability of drawing two non-striped shirts the same for this bag? Why?
9. In a box of 12 batteries, it is known that 5 are dead. Four batteries are selected at random. Find the probability that ...
 - (a) Exactly one dead battery is selected.
 - (b) All four of the selected batteries are dead.
 - (c) *At most two* of the selected batteries are dead.
10. A five-card hand is dealt from an ordinary deck of 52 cards. Find the probability of each hand. Express your answer in symbolic form.
 - (a) Four aces.
 - (b) Four of a kind. (Aces, or Kings, or twos, or... how many are there?)
 - (c) Exactly two hearts.
 - (d) At least three 10's.
 - (e) Three hearts and two spades.
 - (f) Three of one suit and two of another suit. (An example of this would be three hearts and two spades.)
11. There are two jars. The first jar contains 3 red, 2 blue, and 4 white marbles. The second jar contains 4 red, 3 blue, and 2 white marbles. A jar is chosen at random and 3 marbles are drawn. Find the probability that the three marbles are:

- (a) all white
- (b) all red
- (c) all blue
- (d) all the same color

Tough Stuff

12. In row 7 of Pascal's Triangle, the numbers 7, 21, and 35 appear consecutively. Interestingly, these three numbers are in arithmetic sequence. Does this ever happen again? If so, find the next three times it happens. If not, prove it can't happen again.
13. In poker, there is a structure of hand strength (what beats what). Here it is, from best to worst:
- Straight flush: five cards of consecutive rank and suit. For example, **7c 8c 9c Tc Jc** is a straight flush.
 - Four of a kind: four cards of the same rank. For example, **7h 7c 7d 7s Ad** is a four of a kind hand.
 - Full house: three of one rank, and two of another. For example, **2c 2d 2h Jh Js** is a full house ("twos over jacks").
 - Flush: five cards of same suit but not consecutive ranks. For example, **Tc Jc Qc Kc 5c** is a flush (all clubs) and very close to a straight flush.
 - Straight: five cards of consecutive ranks but not the same suit. For example, **Ac 2d 3s 4c 5h** is a straight, ace through five.
 - Three of a kind: Three cards of same rank, other cards of different ranks. For example, **Ks Kc Kd 6h 3s** is a three of a kind hand.
 - Two pair: two pairs of different ranks, and one other card. For example, **6c 6s Td Ts Ks** is a two pair hand.
 - One pair: two cards of same rank, other random cards. For example, **Ac Ah 3s 4d 5s** is a one pair hand.
 - High card: None of the above. For example, **As Ks Qs Jc 9s** is a high card hand, since it has no pairs and is not a straight or flush.

So, the question: if you're dealt five cards, how many ways are there to get each of these things? Is the hand strength correct, from hardest to easiest? For example, is it really harder to get three of a kind than to get two pair?

One way to write suits is as a lowercase letter. In this problem, 7h means the seven of hearts, Tc means the ten of clubs, Qs means the queen of spades, and Ad means the ace of diamonds.

A friend refers to a full house as an "Uncle Jesse" in honor of the ABC sitcom. Oy.

Aces can be high or low (so TJQKA is a straight) but no wraparound is allowed (so QKA23 is not a straight.)

14. What, you want another problem? Okay. Prove that if p is a prime number, then $\binom{pa}{pb}$ has the same remainder as $\binom{a}{b}$ when you divide by p . For example, take $\binom{5}{2}$. Multiply the top and bottom by any prime (say, 7), and you'll get something with the same remainder when you divide by that prime. Using notation, you'd say $\binom{5}{2} = \binom{35}{14}$ modulo 7.

6 Recitation Day

Today is our last day, and it is time to tell each other what we have done. We will start with thirty minutes in which you can think about our problems, especially the lock problem. After that we will ask for volunteers to present solutions, or even partial solutions. These volunteers can come from the teachers and the staff.

Some books on combinatorics

If you want to learn more about combinatorics, here are two books you will find interesting.

- *Proofs that Really Count – The Art of Combinatorial Proof* by Arthur T. Benjamin and Jennifer J. Quinn, published by The Mathematical Association of America.
- *Combinatorics: a Problem Oriented Approach* by Daniel A. Marcus (ISBN 0-88385-710-3)

More problems

Here are some more problems. Perhaps you would like to work on them today, but our main purpose in presenting them to you is that we did not want you to go cold turkey on problem solving. Work on them at your leisure. The staff has agreed to answer email requests for more information or discussion of these problems. Our email addresses are:

- John Polking, polking@rice.edu
- Josef Sifuentes, josefs@rice.edu
- Landon Jennings, landon@rice.edu
- Jason Gershman, jgersh@rice.edu

Binomials

1. Expand (i.e., write without parentheses) each of the following.

(a) $(a + b)^2$

- (b) $(a + b)^3$
- (c) $(a + b)^4$
- (d) $(a + b)^5$

2. What patterns do you observe in Problem 1? Use some triangle thing to expand $(a+b)^6$ without actually multiplying everything out.
3. The binomial theorem (or binomial expansion) says that

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

Can you see why this is true using your knowledge of permutations and combinations?

4. Use a calculator to expand $(0.25r + 0.75w)^5$.
5. Expand $(M + N)^4$ where $M = 2d$ and $N = 7$. In other words, expand $(2d + 7)^4$ without a calculator, but we ask it the weird way for a reason.
6. You take an exam in Japanese with five multiple-choice questions. Each question has four possible answers, and one is right. The only problem is — you don't know any Japanese, so you're stuck making complete and utter random guesses.
 - (a) Find the probability of getting all five questions right.
 - (b) Find the probability of getting all five questions wrong.
 - (c) Find the probability of getting exactly two right.
 - (d) Is it more likely for you to get two questions right, or three questions right? Explain how you know.
7. On a ten-question true-or-false test, how many different *ways* are there to answer the test and get exactly seven questions right? Is there a notation for this?
8. Use some triangle thingy to find the number of different ways there are to answer a ten-question true-or-false test and get at least seven questions right.
9. What is the sum of the numbers in the 10th row of Pascal's Δ ? How is this related to a ten-question true-or-false test?
10. Find the first five powers of 99, and explain what is happening using the Binomial Theorem or Pascal's Triangle... which are basically the same thing.

Check it out, it's a binomial...

Lattices and linear equations

15. Does the line $4x + 7y = 22$ contain any lattice points in Quadrant I of the coordinate plane?
16. For what positive integers N does the line $4x + 7y = N$ *not* contain any lattice points in Quadrant I? (For the purposes of this problem, a point on either axis *is* considered to be part of Quadrant I.)

Say again?

Lattice point: a point with integer coordinates, like (5,11).

Quadrant I: the zone where both x and y are positive.