Transformations of the Plane

A *transformation* of the Euclidean plane is a rule that assigns to each point in the plane another point in the plane. Transformations are also called *mappings*, or just *maps*. Examples of transformations are reflections, rotations, and translations. We will see others in what follows.

It will be our purpose here to learn more about mappings of the plane. However, another goal will be to learn how mathematicians work. You will be expected to work like mathematicians do in order to understand concepts. In particular they work toward definitions, and they use proof to derive new facts. Let's say a little more about each of these.

Definition

We all have an idea of what a definition is. Mathematicians, however, carry the concept to an extreme. They work with an idea until they can express it precisely and without misunderstanding. When approaching a new mathematical concept we all find our own way of thinking about it — we make our own definition. This is not sufficient for mathematicians. They work with the idea, through thought and conversation with others, until all agree on the formal definition they finally adopt. This provides a firm basis for future work. In these sessions we will expect you to do the same.

Proof

A proof is a logical argument demonstrating the truth of a statement. It starts with accepted facts and logically proceeds to the statement itself. One example of a proof is a two column proof that you often see in geometry course. However, that is not the only kind of proof with which you are familiar. Another example is an algebraic derivation. The result of such a proof is an algebraic equation which can now be accepted as a true statement. Most proofs are expressed in paragraph form, using sentence structure. They could perhaps be expressed in two column form, but we do not encourage that.

In these sessions you will be asked a large number of questions. In each case you are expected to provide a proof in the sense we have just explained. Your instructors will frequently respond to a conversation with you with the question, "Why?" This means that they want you to give a logical explanation for what you have just said — a proof.

Notation and Terminology

The Euclidean plane is denoted by $\mathcal{E}$.

We will denote points in the plane $\mathcal{E}$ by capital letters like A and B or by lower case letters such as p and q or x and y. The distance between the points A and B will be denoted by AB.

We will use letters such as l, m, and n to denote lines in the plane. The line through two points p and q will be denoted by $\overleftrightarrow{pq}$. The line segment joining p and q is $\overline{pq}$. The ray or half-line starting at p and proceeding indefinitely through q is $\overrightarrow{pq}$.

Given three distinct points A, B, and C, we will denote the directed angle at B which starts at $\overrightarrow{BA}$ and proceeds to $\overrightarrow{BC}$ by $\angle ABC$. We will only use directed angles, so there should be no confusion. We will use $\theta = \mathrm{m}\angle ABC$ to indicate the measure of the angle in radians.

The triangle with vertices A, B, and C is denoted by $\triangle ABC$. The quadrilateral with vertices a, b, c, and d and sides $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, and $\overline{da}$ is denoted by $\square\, abcd$.

1 Reflections

Hey, welcome to the class. We know you'll learn a lot of math here—maybe some new tricks, maybe some new perspectives on things you're already familiar with. A few things you should know about how the class is organized:

- You do not have to answer all of the questions, ever. If you're answering every question, we haven't written the problem sets correctly.
- You are not required to get to a certain problem number. Some participants might spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- Have fun! Make sure you're spending time working on problems you're interested in. Feel free to skip problems that you're already sure of. Relax and enjoy!
- Each day starts with problems like the ones below, intended to be picked up on regardless of how much or little work you've done on prior sets. Try them as a starting point.
- *Whatever you do, do well.* Flying through the problem set helps no one, especially yourself—you're going to miss the big ideas that others are grabbing onto! There is more to be found in these problems than their answers.

Needed Stuff

Okay, let's get started. We will start with some easy things.

Mirror symmetry

One of our goals is to reach formal definitions of mirror symmetry, and of a reflection across a line. In order to do that you will be given a number of questions. Answering those questions will increase your understanding, and enable you to come to a definition. Let's start with some easy stuff, largely a reprise on the earlier session.

1. Find all of the lines of symmetry for each of the following letters.

A P O W Z

2. How many lines of symmetry does a perfect circle have?

Reflections

Related to mirror symmetry across a line l there is a transformation of the plane called the *reflection across* l. The reflection through l can be visualized by imagining the plane as a piece of paper with the line drawn on it. Then rotate the paper in space 180° about the line.

3. Give a definition of a reflection.

Don't worry too much about your answer. We will give you many chances to change your mind. In fact you can change it any time you want. After writing down your answer discuss it with the others at your table, and then proceed. You will have to provide answers to the following questions based on your definition.

In Figure 1 A is a point on the line l, and B is a point not on l. B' is the reflection of B across l, and X is the intersection of the segment BB' and l.

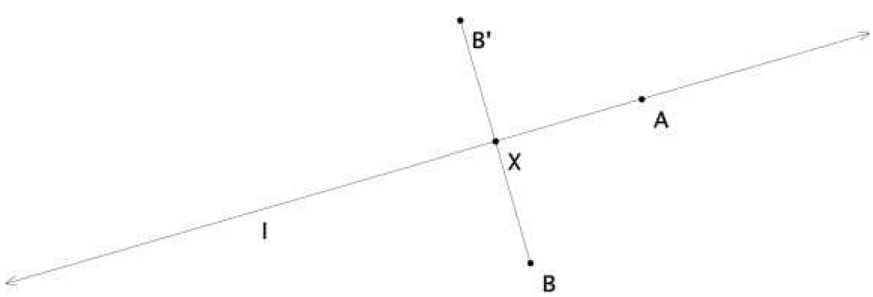

Figure 1: Reflection across the line l.

4. What is the image of A?

5. What can you say about the angles at X?

6. Compare the distances BX and $B'X$.

7. What is the relationship between the line segment $\overline{BB'}$ and the line l?

New Stuff

8. Compare the triangles $\triangle BAX$ and $\triangle B'AX$ in Figure 1.

9. If Y is another point that is not on l, how would you construct the image of Y?

After the last few problems you can see that the reflection across the line l is really a function that sends points in the plane to points in the plane. We will denote this function by M_l. (The M stands for mirror, and the subscript is the line l.) The *domain* of this function is the entire plane $\mathcal{E}$. For a point A in the plane, the image is denoted by $M_l(A)$. Using this notation, the answer to Problem 4 is $M_l(A) = A$ if $A \in l$.

We will call a function which send points of $\mathcal{E}$ to other points of $\mathcal{E}$ a *transformation* of $\mathcal{E}$. We will also call it a *mapping* or simply a *map*. We will say that a transformation *maps* the plane into itself.

It is important to remember that a function is a rule that assigns to any point another point. This means that to define the function you must provide the rule.

With that behind us, let's return to Problem 3.

10. Give a precise mathematical definition of a reflection. (*Hint:* Look back at Problems 4 and 9.) The most precise way would be to complete the equation
$$M_l(A) =$$

Properties of Transformations

For a set $S \subset \mathcal{E}$, the *image* of S under the reflection M_l is the set consisting of all of the images of the points in S. Using set theoretic notation,

$$M_l(S) = \{\, M_l(x) \mid x \in S \,\}.$$

11. What is $M_l(l)$?

12. Show that l is a line of symmetry for a set S if and only if $M_l(S) = S$. Can you use this fact to give a precise, mathematical definition of a line of symmetry for a set S?

We will frequently give our own, preliminary versions of definitions. Here is the first one.

Definition 1: An *isometry* is a transformation of the plane which preserves distances. This means that if A and B are any two points in the plane, with images A' and B' then the distance between A' and B' is the same as the distance between A and B. In symbols, $A'B' = AB$.

13. Show that a reflection is an isometry. (*Hint:* Consider two cases. First assume the points A and B are on the same side of the axis of reflection, and then assume they are on different sides of the axis.)

14. If $X' = M_l(X)$, what is $M_l(X')$?
15. Suppose that A, B, and C are three distinct points in $\mathcal{E}$, and set $A' = M_l(A)$, $B' = M_l(B)$, and $C' = M_l(C)$. How do the triangles $\triangle ABC$ and $\triangle A'B'C'$ compare?

In the next couple of problems we will define properties that some transformations and functions have. You can illustrate these properties using reflections.

16. Is a reflection M_l a *one-to-one* function? That is, if x and y are distinct points in $\mathcal{E}$ are the points $M_l(x)$ and $M_l(y)$ also distinct?
17. Identify the *range* of the reflection M_l. The range of M_l is the set $M_l(\mathcal{E})$.

Next we want to consider a different kind of transformation. A *fold* along a line l keeps one side of the line fixed, and folds the other side over the line. A fold is quite easily visualized using a piece of patty paper.

18. If A is a point on the fixed side of l, what is its image under a fold along l?
19. If A is a point on the other side of l, what is its image under a fold along l?
20. Is a fold an isometry?
21. Is a fold one-to-one?
22. What is the range of a fold?

Tough Stuff

This section contains problems that a somewhat harder than the earlier problems. They may preview coming material and may go in a different direction entirely.

Consider the rectangle $\square ABCD$ in Figure 2. It has two lines of symmetry, l and k. What is the effect of reflecting first about l and then about k? The result of applying one transformation after another is a new transformation called the *composition* of the transformations. If F is the composition, then using function notation this is written as

$$F = M_k \circ M_l,$$

and by definition

$$F(X) = M_k \circ M_l(X) = M_k(M_l(X)).$$

We will ask you to identify the composition of M_k and M_l. We will approach an answer to this problem in several steps. Let $F = M_k \circ M_l$.

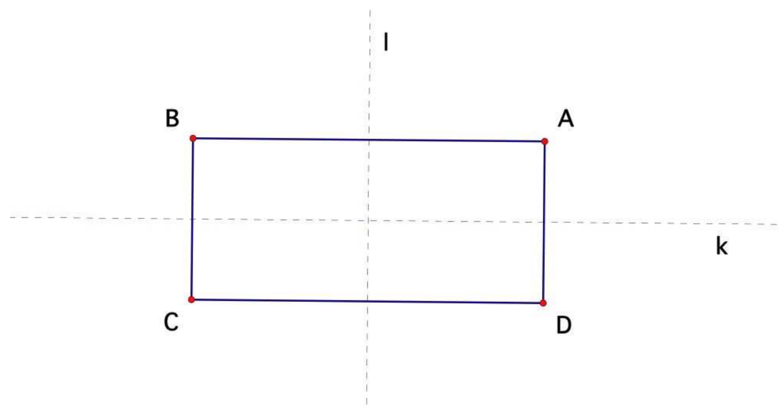

Figure 2: A rectangle with two lines of symmetry.

23. Referring to Figure 2, what is $F(A)$? Identify the images of the rest of the vertices.

24. A point X is a *fixed point* of a transformation G if $G(X) = X$. Identify the fixed points of the reflections M_l, and M_k.

25. Find the fixed points of $F = M_k \circ M_l$. (*Hint:* If $F(X) = X$, show that $M_k(X) = M_l(X)$. Use the definition of a reflection to find which points can satisfy this equation.)

26. Is $F = M_k \circ M_l$ a reflection?

27. Can you identify the transformation $F = M_k \circ M_l$?

28. Show that if $X \in \mathcal{E}$ and l is any line, then $M_l \circ M_l(X) = X$. Thus the transformation $M_l \circ M_l$ fixes every point in the plane. It will be useful to have a term and a symbol for the transformation that fixes every point. We will call it the *identity* map, and denote it by I. Thus $I(X) = X$ for every point X. We can phrase the result of this problem by saying that $M_l \circ M_l = I$.

Next we will consider two lines l and k intersecting at the point C with an angle θ, as illustrated in Figure 3, and we will want to analyze the composition $F = M_k \circ M_l$. The angle θ is a directed angle measured in the counter-clockwise direction.

29. Find the fixed points of F.

30. Suppose that $X \neq C$. Let $X' = M_l(X)$, and $X'' = M_k(X') = M_k \circ M_l(X) = F(X)$ (see Figure 3). Show that $CX = CX' = CX''$.

31. With X chosen as in Figure 3. Show that $\mathrm{m}\angle XCX'' = 2\theta$.

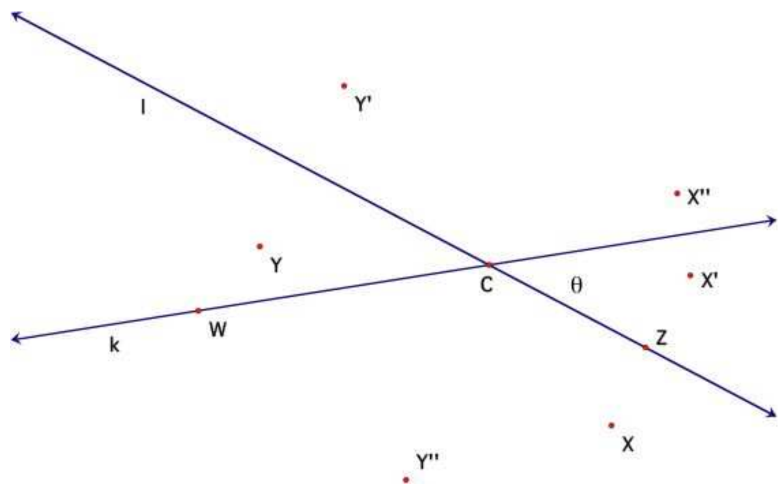

Figure 3: Composition of the reflections across two intersecting lines.

32. Notice from Problem 31 that m$\angle XCX''$ is independent of the point X. It is likely that your proof depends on the position of X relative to the lines l and k as shown in Figure 3. Can you prove the same thing for m$\angle YCY''$, where Y is positioned as shown in Figure 3, and $Y'' = F(Y)$? How about m$\angle WCF(W)$ and m$\angle ZCF(Z)$?

33. Can you give a concise description of the composition $F = M_k \circ M_l$?

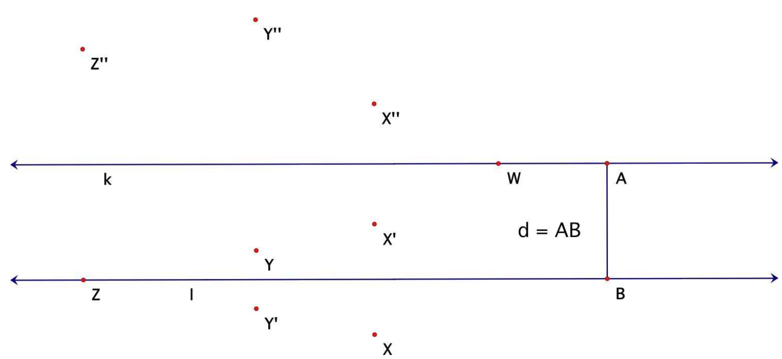

Figure 4: Composition of the reflections across two parallel lines.

In Figure 4 we see two parallel lines l and k. The segment $\overline{AB}$ is perpendicular to both, so $d = AB$ is the distance between these lines. We want to examine the composition $F = M_k \circ M_l$. For the point X, we have $X' = M_l(X)$ and $X'' = M_k(X') = f(X)$. Similarly $Y'' = F(Y)$ and $Z'' = F(Z)$.

34. With X positioned as in Figure 4, show that $XX'' = 2d$.

35. Problem 19 shows that the distance $XF(X)$ is independent of the point X, at least if X is situated below the line l, as we see it in Figure 4. Can you prove that $YF(Y) = 2d$, if Y is situated between the lines as indicated in Figure 4? What about $ZF(Z)$ and $WF(W)$, if $Z \in l$ and $W \in k$?

36. Show that $\overline{Xf(X)}$ is perpendicular to l and to k for every point X.

37. What is special about the quadrilateral $\square XYF(Y)F(X)$, where X and Y are any two points?

38. Assuming that l and k are parallel, but different, find the fixed points of $F = M_k \circ M_l$.

39. Can you give a concise description of the composition $F = M_k \circ M_l$?

2 Rotations and Translations

Well, we know quite a lot about reflections, but there are other transformations called rotations and reflections. We assume you know a little about them, but it is important that we are all on the same page, so we will provide explicit definitions

Important and Easy Stuff

Reflections

Remember that the reflection about the line l is the map defined by

$$M_l(A) = \begin{cases} A, & \text{if } A \in l, \\ A', & \text{if } A \notin l \text{ and } l \text{ is the perpendicular bisector of } \overline{AA'}. \end{cases}$$

Rotations

You have had some experience with rotations. You know that a rotation is a transformation which has a *center* C, which is a fixed point and an *angle of rotation* θ. If X is a point which is not the center, then the image of X

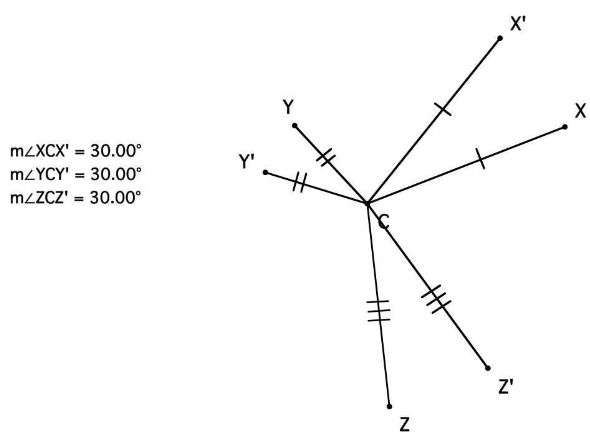

Figure 1: The effect of the rotation $R_{C,30^\circ}$ on the points X, Y, and Z.

under the rotation is the point X' for which the angle $m\angle XCX' = \theta$, and the distance $CX' = CX$. Thus a rotation is determined by its center C, and θ, the angle of rotation. We will denote the rotation with center C and rotation angle θ by $R_{C,\theta}$. Figure 1 illustrates a rotation with a 30° rotation angle.

In the definition of rotation, θ is a directed angle. This means that it is positive in the counter-clockwise direction and negative in the clockwise direction. This is illustrated in Figure 2.

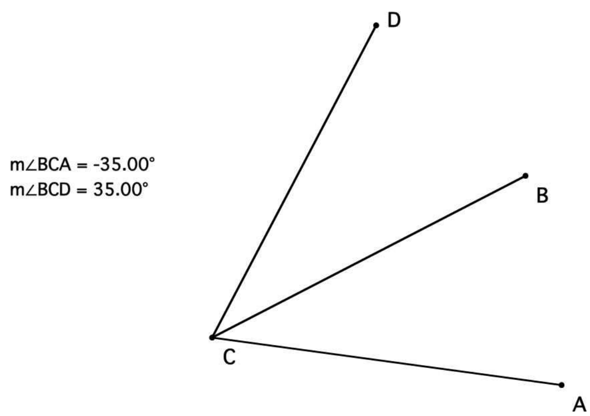

Figure 2: Examples of directed angles.

Rotational Symmetry

Rotational symmetries of a set are defined very much like mirror symmetries. We say that a set $S \subset \mathcal{E}$ has a rotational symmetry with center C and angle θ if $R_{C,\theta}(S) = S$.

Figure 3: A rectangle has rotational symmetries

1. Find all rotational symmetries of the rectangle in Figure 3.
2. Find all rotational symmetries of the square in Figure 4.
3. Find all rotational symmetries of the equilateral triangle in Figure 5.

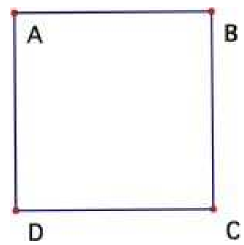

Figure 4: A square has more rotational symmetries

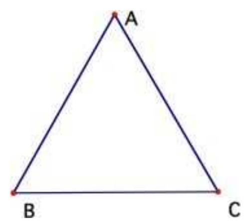

Figure 5: An equilateral triangle has rotational symmetries

4. Find all rotational symmetries of the letters

O S V X Z

Translations

Let A and B be two distinct points in the plane. The *vector* $\mathbf{v}$ with base A and tip B is shown in Figure 6. If C and D are two other points such

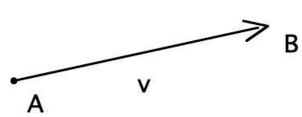

Figure 6: The vector $\mathbf{v}$ with base A and tip B.

that $\square BACD$ is a parallelogram, and $\mathbf{w}$ is the vector with base C and tip D, the vectors $\mathbf{v}$ and $\mathbf{w}$ are considered to be *equivalent*, since they have the same length and the same direction. See Figure 7.

5. Suppose A, B, C, and D are collinear. Then $\square BACD$ degenerates and cannot be a parallelogram. Yet in some cases the vectors $\mathbf{v}$ and

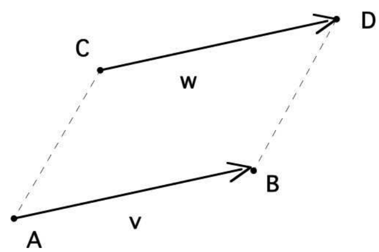

Figure 7: The vectors **v** and **w** are equivalent.

w will have the same length and direction, so we will want them to be equivalent. How do we specify when **v** and **w** are equivalent?

Given a vector **v**, the *translation along* **v** is defined by $T_{\mathbf{v}}(X) = X'$ if the vector with base X and tip X' is equivalent to **v**. The concept is illustrated in Figure 8.

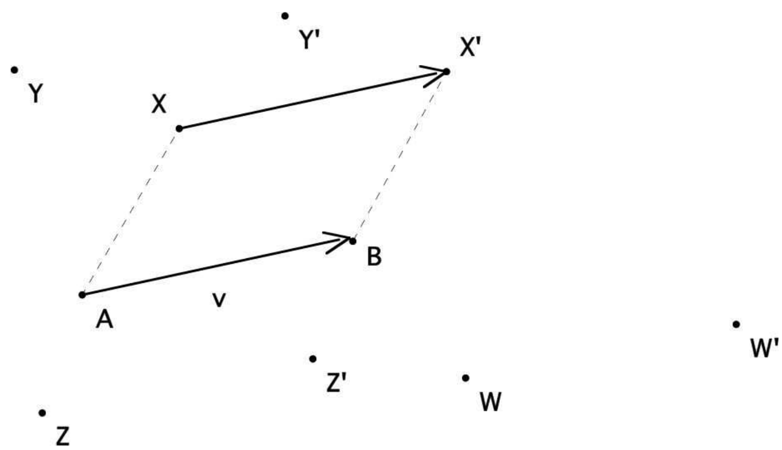

Figure 8: Several points and their translates along the vector **w**.

Translational Symmetry

We will say that a set S has translational symmetry along a vector **v** if $T_{\mathbf{v}}(S) = S$. Notice that this definition is very similar to that of reflection and rotational symmetry.

6. Suppose that A and B are distinct points and **v** is the vector with base A and tip B. Does the line $\overleftrightarrow{AB}$ have translational symmetry along **v**?

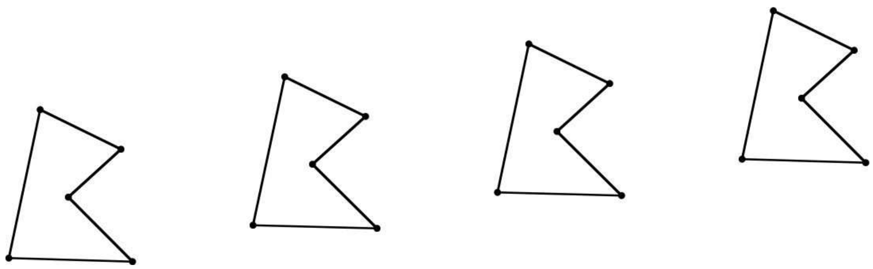

Figure 9: A set with translational symmetry.

7. Consider the set in Figure 9. Assume the pattern repeats infinitely often in each direction. Find a vector **v** along which this set has translational symmetry. Is there more than one such vector? If the set does not repeat infinitely often in each direction does it have translational symmetry?

8. Consider a floor of infinite extent that is tiled with hexagons as shown in Figure 10. Find vectors along which this set has translational symmetry.

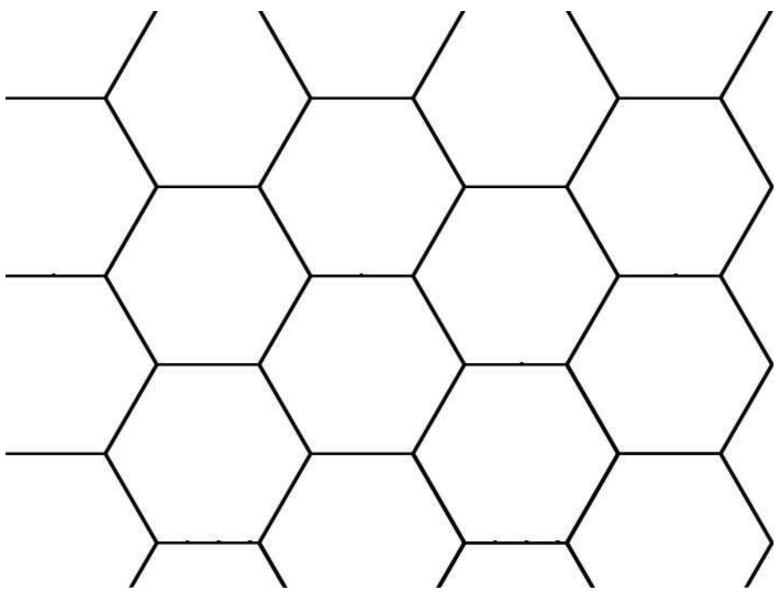

Figure 10: A floor tiled with hexagons.

New Stuff

Some of you did this yesterday, at least up to Problem 16, so you should be able to get through it quickly. We will add more problems to make up for that.

9. Show that if $X \in \mathcal{E}$ and l is any line, then $M_l \circ M_l(X) = X$. Thus the transformation $M_l \circ M_l$ fixes every point in the plane. It will be useful to have a term and a symbol for the transformation that fixes every point. We will call it the *identity* map, and denote it by I. Thus $I(X) = X$ for every point X. We can phrase the result of this problem by saying that $M_l \circ M_l = I$.

We might call the identity map the lazy map, since it does nothing.

10. A transformation F is said to be *invertible* if there is another transformation G such that $G \circ F = I$ and $F \circ G = I$. The map G is then called the *inverse* of F, and it is denoted by F^{-1}. Problem 9 can be interpreted to say that $M_l^{-1} = M_l$. What is the inverse of the rotation $R_{C,\theta}$?

11. What it the inverse of the translation $T_{\mathbf{v}}$?

Consider two lines l and k intersecting at the point C with an angle θ, as illustrated in Figure 11. The angle θ is a directed angle measured in the counter-clockwise direction. What is the effect of reflecting first about l and then about k? The result of applying one transformation after another is a new transformation called the *composition* of the transformations. If F is the composition, then using function notation this is written as

$$F = M_k \circ M_l,$$

and by definition

$$F(X) = M_k \circ M_l(X) = M_k(M_l(X)).$$

We will want to analyze the composition $F = M_k \circ M_l$.

12. Find the fixed points of F. (*Hint:* This is not so easy. However, you might apply the reflection M_k and the result of Problem 9 to show that the equation $F(X) = X$ implies that $M_l(X) = M_k(X)$. Then ask when this equation can be true.)

13. Suppose that $X \neq C$. Let $X' = M_l(X)$, and $X'' = M_k(X') = M_k(M_l(X)) = F(X)$ (see Figure 11). Show that $CX = CX' = CX''$.

14. With X chosen as in Figure 11 show that $\mathrm{m}\angle\, XCX'' = 2\theta$.

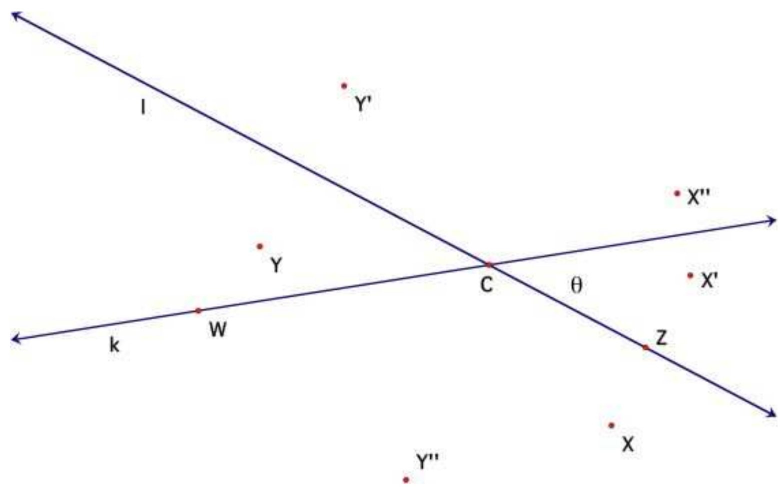

Figure 11: The composition of two reflections across intersecting lines.

15. Notice from Problem 14 that $m\angle XCX''$ is independent of the point X. It is likely that your proof depends on the position of X relative to the lines l and k as shown in Figure 11. Can you prove the same thing for $m\angle YCY'$, where Y is positioned as shown in Figure 11, and $Y'' = F(Y)$? How about $m\angle WCF(W)$ and $m\angle ZC(F(Z)$?

16. Can you prove that $F = M_k \circ M_l = R_{C,2\theta}$?

17. Suppose we have a rotation $R_{C,\theta}$. Find two lines l and k such that $R_{C,\theta} = M_k \circ M_l$. How much flexibility do you have in choosing l and k?

If you have gotten through the last two problems, then you have proved an important and interesting theorem. Here it is.

Theorem.

(a) *Suppose that l and k are two distinct lines that intersect at the point C with the angle from l to k being θ. Then*

$$M_k \circ M_l = R_{C,2\theta}.$$

(b) *Consider the rotation $R_{C,\theta}$. If l and k are any two lines which intersect at C with directed angle from l to k equal to $\theta/2$ then*

$$R_{C,\theta} = M_k \circ M_l.$$

18. Show that a rotation is an isometry.

In Figure 12 we see two parallel lines l and k. The segment $\overline{AB}$ is perpendicular to both, so $d = AB$ is the distance between these lines. We want to examine the composition $F = M_k \circ M_l$. For the point X, we have $X' = M_l(X)$ and $X'' = M_k(X') = F(X)$. Similarly $Y'' = F(Y)$ and $Z'' = F(Z)$.

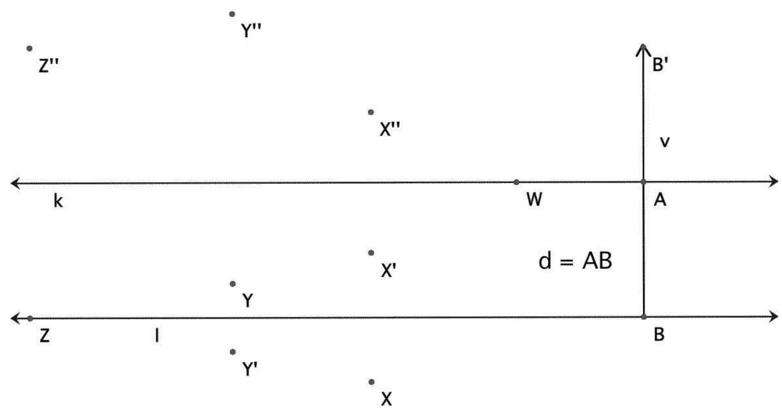

Figure 12: The composition of two reflections across parallel lines.

19. With X positioned as in Figure 12, show that $XX'' = 2d$.

20. Problem 19 shows that the distance $XF(X)$ is independent of the point X, at least if X is situated below the line l, as we see it in Figure 12. Can you prove that $YF(Y) = 2d$, if Y is situated between the lines as indicated in Figure 12? What about $ZF(Z)$ and $WF(W)$, if $Z \in l$ and $W \in k$?

21. Show that $\overline{Xf(X)}$ is perpendicular to l and to k for every point X.

22. What is special about the quadrilateral $\square XYF(Y)F(X)$, where X and Y are any two points?

23. Let B' be the reflection of B across the line k, and let $\mathbf{v}$ be the vector with base B and tip B'. Show that $F = M_k \circ M_l = T_{\mathbf{v}}$.

24. Suppose we have a translation $T_{\mathbf{v}}$. Find two lines l and k such that $T_{\mathbf{v}} = M_k \circ M_l$. How much flexibility do you have in choosing lines l and k?

If you have gotten through the last two problems, then you have proved another important and interesting theorem. Here it is.

Theorem.

(a) *Suppose that l and k are two distinct parallel lines. Let B be a point on l and let B' be the reflection of B across k. If $\mathbf{v}$ is the vector with base B and tip B', then*

$$M_k \circ M_l = T_{\mathbf{v}}.$$

(b) *Consider the translation $T_{\mathbf{v}}$. Let B be any point in the plane and choose B' such that $\mathbf{v}$ is equivalent to the vector with base at B and tip at B'. Let A be the midpoint of $\overline{BB'}$. Let l be the line through B perpendicular to $\overline{BB'}$ and let k be the line through A perpendicular to $\overline{BB'}$. Then*

$$T_{\mathbf{v}} = M_k \circ M_l.$$

25. Show that any translation is an isometry.

26. Assuming that l and k are parallel, but different, find the fixed points of $F = M_k \circ M_l$.

Tough Stuff

Here we will begin to examine the properties of isometries. Remember that an isometry is a transformation of the plane which preserves the distance between points. Thus if F is an isometry, then the distance between $F(A)$ and $F(B)$ is the same as the distance between A and B. Further, this is true for any choices of A and B.

We already know that reflections, rotations, and translations are isometries. One of our goals is to discover if there are any other isometries.

27. Show that the identity map I is an isometry.

28. Show that any isometry is one-to-one.

29. Show that any isometry maps line segments into line segments. More precisely, if F is an isometry and A and B are points, then

$$F(\overline{AB}) = \overline{F(A)F(B)}\ .$$

Don't forget to show that every point of $\overline{F(A)F(B)}$ is the image of a point in $\overline{AB}$.

30. Any isometry maps lines into lines. More precisely, if F is an isometry, and if $l = \overleftrightarrow{AB}$, where A and B are distinct points, then

$$F(l) = F(\overleftrightarrow{AB}) = \overleftrightarrow{F(A)F(B)}.$$

31. Any isometry maps rays into rays. More precisely, if F is an isometry, and A and B are distinct points, then

$$F(l) = F(\overrightarrow{AB}) = \overrightarrow{F(A)F(B)}.$$

32. Suppose that F is an isometry and A, B, and C are three non-collinear points. Show that the triangle $\triangle F(A)F(B)F(C)$ is congruent to $\triangle ABC$.

33. Any isometry maps an angle onto a congruent angle.

34. Any isometry sends two perpendicular lines into two perpendicular lines.

35. Any isometry sends parallel lines into parallel lines.

36. Any isometry sends a parallelogram into a parallelogram.

37. Show that any isometry is onto the plane $\mathcal{E}$. In other words, if F is an isometry and Y is any point, then there is a point X such that $F(X) = Y$.

38. Since an isometry F is one-to-one and onto, it has an inverse F^{-1}. Show that F^{-1} is also an isometry.

39. Show that any isometry maps circles onto circles. More precisely, if F is an isometry and $\mathcal{C}$ is the circle with center C and radius r, then $F(\mathcal{C})$ is the circle with center $F(C)$ and radius r.

40. Prove the following Theorem.

    Theorem (Triangle Isometry Theorem)**.** *Suppose that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Show that there is an isometry F composed of three or fewer reflections such that $F(A) = A'$, $F(B) = B'$, and $F(C) = C'$.*

    (*Hint:* Try to construct the isometry as the composition of three or fewer reflections, each of which is intended to move one vertex onto its counterpart.)

41. Suppose the isometry F has two distance fixed points A and B. Show that every point on the line $\overleftrightarrow{AB}$ is a fixed point.

42. Suppose the isometry F has three non-collinear fixed points. Show that F is the identity.

43. Suppose that F and G are isometries, and that there are three non-collinear points A, B, and C such that $F(A) = G(A)$, $F(B) = G(B)$, and $F(C) = G(C)$. Show that $F = G$. (*Hint:* What are the fixed points of $F^{-1} \circ G$?)

44. Prove the following theorem.

Theorem (Three Reflections Theorem)**.** *Any isometry of the plane can be expressed as the composition of at most three reflections.*

3 Isometries

We have learned a lot about transformations of the plane. We understand reflections, rotations, and translations. In particular we know that all of these are isometries. Remember that a transformation F is an isometry if it preserves distances. Thus, if A and B are arbitrary points, then $F(A)F(B) = AB$. Now we will look at isometries on their own.

Think about what an isometry does. It moves the plane about without disturbing the distances between points. Another name for an isometry is *rigid motion.* Can you see why that is a good name for an isometry?

One of our goals will be to classify isometries. We want to know what all of them are. For example, is every isometry a reflection, a rotation, or a translation? Are there any other types of isometries?

We will start by reviewing what we already know.

Stuff We Know

Reflections

We started by looking at reflections. The reflection across the line l is the transformation defined by

$$M_l(X) = \begin{cases} X & \text{if } X \in l, \\ X' & \text{if } X \notin l \text{ and } l \text{ is the perpendicular bisector of } \overline{XX'}. \end{cases}$$

We proved that a reflection is an isometry, and that

$$M_l \circ M_l = I, \tag{1}$$

where I is the identity map.

Rotations

The rotation with center C and angle of rotation θ is the transformation defined by

$$R_{C,\theta}(X) = X' \quad \text{where } CX' = CX \text{ and } \mathrm{m}\angle XCX' = \theta.$$

Rotations and reflections are related according to the following theorem, which we have already proved.

Theorem (Reflections through Intersecting Lines)**.**

(a) *Suppose that l and k are two distinct lines that intersect at the point C with the angle from l to k being θ. Then*

$$M_k \circ M_l = R_{C,2\theta}.$$

(b) *Consider the rotation $R_{C,\theta}$. If l and k are any two lines which intersect at C with directed angle from l to k equal to $\theta/2$ then*

$$R_{C,\theta} = M_k \circ M_l.$$

Translations

Suppose that A and B are two distinct points, and let $\mathbf{v}$ denote the vector with base A and tip B. The translation along the vector $\mathbf{v}$ is the map

$T_{\mathbf{v}}(X) = X'$ where the vector with base X and tip X' is equivalent to $\mathbf{v}$.

Translations and reflections are related by the following theorem, which we have already proved.

Theorem (Reflections through Parallel Lines)**.**

(a) *Suppose that l and k are two distinct parallel lines. Let B be a point on l and let B' be the reflection of B across k. If $\mathbf{v}$ is the vector with base B and tip B', then*

$$M_k \circ M_l = T_{\mathbf{v}}.$$

(b) *Consider the translation $T_{\mathbf{v}}$. Let B be any point in the plane and let $B' = T_{\mathbf{v}}(B)$. Let A be the midpoint of $\overline{BB'}$. Let l be the line through B perpendicular to $\overline{BB'}$ and let k be the line through A perpendicular to $\overline{BB'}$. Then*

$$T_{\mathbf{v}} = M_k \circ M_l.$$

New Stuff

The Triangle Isometry Theorem

We already know what can happen if we look at the composition of two reflections. What can we do with three reflections? The beginning of the answer is in the following theorem.

Theorem (Triangle Isometry)**.** *If the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent then there is an isometry F composed of three or fewer reflections such that $F(A) = A'$, $F(B) = B'$, and $F(C) = C'$.*

You already know everything you need to know to prove this very nice result, but to set you on the right track we will pose a couple of preliminary problems.

1. Suppose that A and A' are two distinct points in the plane. Find a line l so that $M_l(A) = A'$.

2. Suppose that A and A' are two distinct points in the plane. What is the set of points that are equidistant from A and A'?

3. Prove the triangle isometry theorem. (*Hint:* Start by finding a reflection that maps A to A', if necessary. This first reflection is shown in Figure 1. Look closely at what this reflection does to $\triangle ABC$. Then sequentially choose reflections that map the other vertices to their desired images. Draw figures to show the effect of the reflections.)

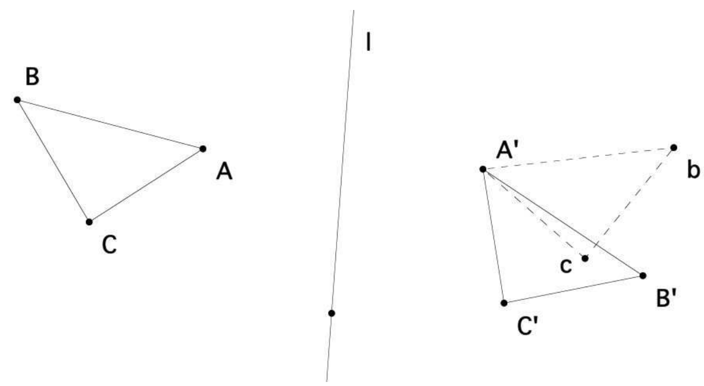

Figure 1: The first reflection maps A to A', B to b, and C to c.

Isometries and the Three Fixed Points Theorem

Now we will turn to isometries. All we know at this point is the definition. A transformation F is an isometry if $F(A)F(B) = AB$ for every pair of points A and B. We will learn a lot more by looking at the fixed points of an isometry. One of our goals is the following theorem.

Theorem (Three Fixed Points)**.** *Suppose the isometry F has three non-collinear fixed points. Then F is the identity map.*

4. Suppose the isometry F has a fixed point P. Suppose that A is not a fixed point, and let $A' = F(A)$. Show that P lies on the perpendicular bisector of the segment $\overline{AA'}$. (See Figure 2.)

This is referred to as the *bisector-fixed point property.* Remember it!

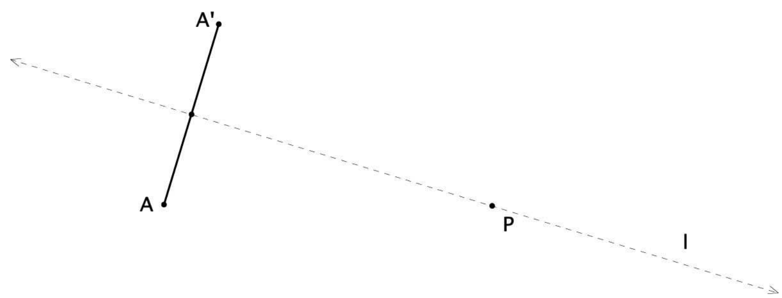

Figure 2: The fixed point P is on the perpendicular bisector of $\overline{AA'}$.

5. Suppose the isometry has two distinct fixed points P and Q. Show that every point on the line $\overleftrightarrow{PQ}$ is a fixed point.

6. Prove the three fixed points theorem.

7. Suppose that the isometry F has at least two distinct fixed points P and Q. Show that F is either the identity or the reflection across the line $l = \overleftrightarrow{PQ}$. (*Hint:* Suppose that $A \notin l$. Either $F(A) = A$ or $F(A) \neq A$. In the latter case remember the bisector-fixed point property and equation (1). Use the three fixed points theorem.)

The Three Reflections Theorem

Our next goal is to prove a result that applies to all isometries.

Theorem (Three Reflections)**.** *Any isometry of the plane can be expressed as the composition of at most three reflections.*

The three reflections theorem tells us a lot about isometries. It means that we do not have to look around for some crazy maps that happen to be isometries. They can all be found by composing those simple maps, reflections.

We will provide a proof through our problems, but first we will need some preliminary results.

8. Suppose that F is an isometry and A, B, and C are three non-collinear points. Let $A' = F(A)$, $B' = F(B)$, and $C' = F(C)$. Show that $\triangle A'B'C' \cong \triangle ABC$.

9. Prove the three reflections theorem. (*Hint:* Use the triangle isometry theorem, and the three fixed points theorem. Your solution to Problem 7 might help as well.)

10. Show that an isometry is onto. That means that if F is an isometry and Q is any point in the plane, then there is a point P in the plane such that $F(P) = Q$.

11. Show that an isometry is one-to-one. This means that if F is an isometry, and A and B are distinct points, then $F(A)$ and $F(B)$ are also distinct.

12. Since an isometry F is one-to-one and onto, it has an inverse, defined by
$$F^{-1}(A) = A' \quad \text{where } F(A') = A.$$
show that F^{-1} is also an isometry.

13. Suppose that F and G are isometries, and A, B, and C are three non-collinear points for which $F(A) = G(A)$, $F(B) = G(B)$, and $F(C) = G(C)$. Show that $F = G$. ($F = G$ means that $F(X) = G(X)$ for every point X in the plane.) (*Hint:* Consider what $G^{-1} \circ F$ does to the three points, and use the three fixed point theorem.)

Interesting Stuff but not Enough Time

We now know that any isometry is the composition of at most three reflections. We know about the composition of two reflections, but what about the composition of three reflections? There are lots of possible ways to combine three reflections, but we will highlight a particular one.

Suppose $\mathbf{v}$ is the vector with base A and tip B, where $A \neq B$. Let $l = \overleftrightarrow{AB}$. The glide reflection along $\mathbf{v}$ is the mapping

$$G_{\mathbf{v}} = M_l \circ T_{\mathbf{v}}.$$

This transformation is illustrated in Figure 3.

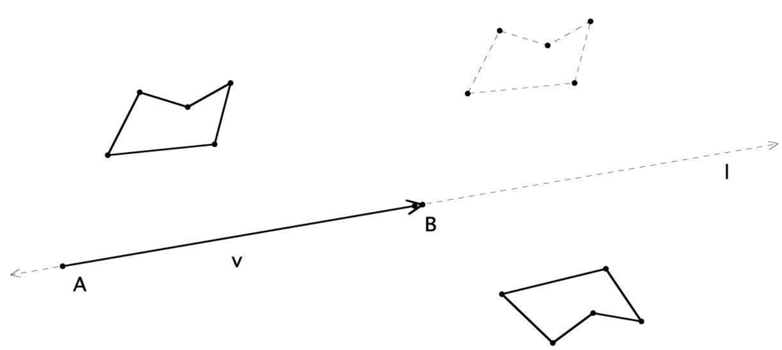

Figure 3: The glide-reflection along the vector $\mathbf{v}$ consists of a translation through $\mathbf{v}$ followed by a reflection across the line l.

The translation $T_{\mathbf{v}}$ is the composition of the reflections through two lines m and k, both of them perpendicular to the vector $\mathbf{v}$ and the line l. The glide reflection $G_{\mathbf{v}}$ is completed by reflecting in the line l. Hence $G_{\mathbf{v}}$

is the composition of three reflections, the first two perpendicular to $\mathbf{v}$ and the third across the line determined by $\mathbf{v}$. This configuration of the lines l, k, and m is shown in Figure 4.

It turns out that when we add glide reflections to our list, we can describe all isometries. The precise statement is contained in the next theorem.

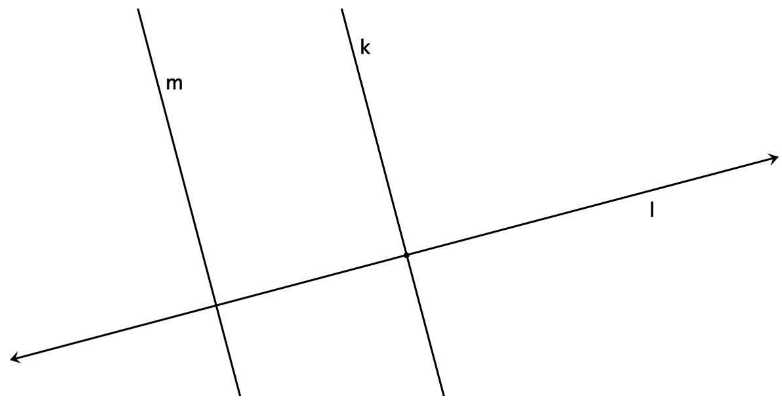

Figure 4: The glide reflection $G_{\mathbf{v}}$ is the composition of the reflections M_l, M_k, and M_m.

Theorem (Classification of Isometries)**.** *Any isometry of the plane is of exactly one of the following types.*

(0) *The identity I.*

(1) *A reflection M_l in a line l.*

(2a) *A rotation $R_{C,\theta}$ of angle $0 < \theta < 2\pi$ about a point C.*

(2b) *A translation $T_{\mathbf{v}}$ along a non-trivial vector $\mathbf{v}$.*

(3) *A glide reflection $G_{\mathbf{v}}$ through the non-trivial vector $\mathbf{v}$.*

In each case the number in the type of the isometry indicates the minimum number of reflections needed to represent that class of isometries.

We have come a long way in the proof of this theorem. We understand transformations that can be expressed as the composition of 0, 1, or 2 reflections. What we need to do is understand the transformations that are expressed as the composition of three reflections. As you will see, the keys to the analysis are reflection through lines theorems, and the fact, expressed in equation (1), that a reflection is its own inverse.

Let's look at some of the easier possibilities. From now on we assume that

$$F = M_l \circ M_k \circ M_m,$$

where l, k, and m are lines. In some cases three reflections are not really necessary.

14. Suppose that the lines l, k, and m intersect in precisely one point P.
    (a) Suppose that $m = k$. Show that $F = M_l$.
    (b) Suppose that $m \neq k$. Show that there is an angle θ such that $F = M_l \circ R_{P,\theta}$. Next show that there is a line m' through P such that $R_{P,\theta} = M_l \circ M_{m'}$ and therefore $F = M_{m'}$.

15. Suppose that the lines l, k, and m are mutually parallel. Show that there is a line m' parallel to these three such that $F = M_{m'}$. (*Hint:* This case is similar to that of the previous problem. Follow the outline there, using a translation instead of a rotation.)

We are left with the case where $F = M_l \circ M_k \circ M_m$ where the lines m, k, and l do not all intersect in a single point, and are not mutually parallel. It will be up to you to show that in this case F is a glide reflection. To do so you need to show that F is the composition of three reflections through lines configured as in Figure 4. To do this, you can use the reflection through intersecting lines theorem.

16. Suppose first that the lines m and k intersect in the point P as shown in Figure 5.

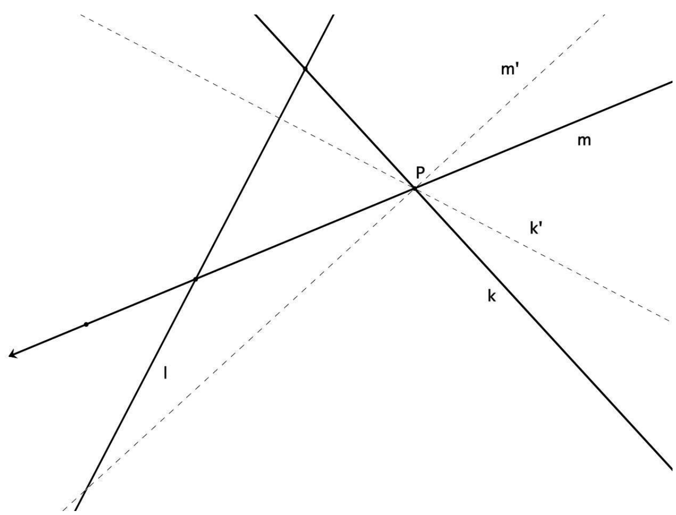

Figure 5: Moving the lines of reflection m and k about the point P where they intersect.

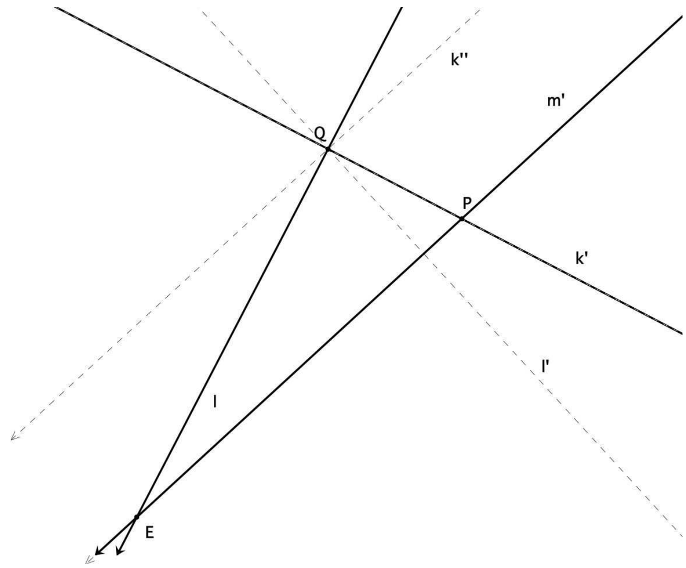

Figure 6: Moving the lines of reflection l and k' about the point Q where they intersect.

(a) Show that there exist lines m' and k' through P such that $k' \perp l$ and $M_{k'} \circ M_{m'} = M_k \circ M_m$, and therefore $F = M_l \circ M_{k'} \circ M_{m'}$.

(b) Let Q be the point of intersection of the lines l and k'. Show that there exist lines l' and k'' through Q such that $l' \perp m'$ and $M_{l'} \circ M_{k''} = M_l \circ M_{k'}$, and therefore $F = M_{l'} \circ M_{k''} \circ M_{m'}$. (See Figure 6.)

(c) Show that F is a glide reflection.

17. In the previous problem we assumed that the lines m and k intersected. Suppose they do not. Show that in this case the lines l and k must intersect. By modifying your solution to the previous problem show that F is a glide reflection in this case as well.